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Hydrodynamic Limits for the Boltzmann Process

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We study the behavior of the nonlinear Markov process associated to the Boltzmann equation under both hyperbolic and parabolic space-time scalings. In the first case the limit of the process is the solution of an o.d.e. with vector field given by a solution of the Euler equation, while in the second case the limit of the process, in the incompressible case, turns out to be a diffusion process whose drift is a solution of the incompressible Navier–Stokes equation.

KEY WORDS: Boltzmann equation; transport processes; hydrodynamic limit; diffusion approximation; stochastic averaging.

1. INTRODUCTION

We consider a gas of hard spheres in the low-density regime on the mean free path scale. It has long been known that the behavior of such a system is described by the Boltzmann equation for the density $f(t, x, v)$ on the one-particle phase space.[1] Lanford[2] proved the convergence of the solution of Liouville equations to the solution of the Boltzmann equation, in the Grad–Boltzmann limit, for short times.

In this paper we take what may be called a Lagrangian point of view, that is, we follow the evolution of a single particle, which we will call the tracer particle. This evolution is stochastic, since we do not keep track of the motion of the other particles. The velocity of our tracer particle jumps as a result of the elastic collisions with the environment particles. The motion of the tracer particle is therefore described by a transport process (X, V) on the phase space $\Pi \times R^3$ (where Π denotes the unit 3-dimensional torus), driven by a jump process inhomogeneous in time and space. The kernel of the generator of the jump process is determined by $f(t, x, v)$ and

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since the tracer particle is identical to the environment particles, the one-dimensional probability distribution of (X, V) has density $f(t, x, v)/\bar{\rho}$, where $\bar{\rho} = \int dx\, dv\, f(t, x, v)$: Thus, (X, V) is a nonlinear Markov process whose forward Kolmogorov equation is the Boltzmann equation. We will call this process the "Boltzmann process." The physical relevance of the Boltzmann process is confirmed by a result of Spohn[3] (see also refs. 4 and 5), who proved the analog of Lanford's theorem for stochastic processes, namely that, at least for short times, (X, V) is the limit of a non-Markov process describing the motion of a tracer particle of the hard-sphere gas on the microscopic scale. The Boltzmann process also underlies some algorithms for the numerical simulation of the Boltzmann equation (see, for instance, refs. 6 and 7).

We are interested in studying the Boltzmann process (X, V) on a hydrodynamic space scale (length unit much larger than the mean free path) and on two different time scales: in fact, it is natural to expect that, if ε is the mean free path, on a time interval of order ε^{-1}, the position component of the Boltzmann process converges, as ε goes to zero, to a deterministic motion, while in order to see any dissipative effect one has to wait for an ε^{-2} time. Therefore we consider the Boltzmann equation under both the hyperbolic rescaling (ε^{-1} in space and ε^{-1} in time)

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \varepsilon^{-1} Q(f^\varepsilon, f^\varepsilon) \tag{1.1}$$

where

$$Q(f, f) = \int_{(v_1 - v)\cdot\omega \geqslant 0} dv_1\, d\omega \\ \times (v_1 - v)\cdot\omega \{ f(t, x, v_1')\, f(t, x, v') - f(t, x, v_1)\, f(t, x, v) \} \tag{1.2}$$

and the parabolic rescaling (ε^{-1} in space and ε^{-2} in time)

$$\partial_t F^\varepsilon + \varepsilon^{-1} v \cdot \nabla_x F^\varepsilon = \varepsilon^{-2} Q(F^\varepsilon, F^\varepsilon) \tag{1.3}$$

For the hyperbolic rescaling, Caflish[8] (see also Lachowicz[9]) proved that for any smooth solution (ρ, T, u) of the compressible Euler equation, with ρ and T bounded away from 0, there exists a solution of (1.1) which differs from the local Maxwellian of parameters (ρ, T, u) by an infinitesimal of the order of ε. For the parabolic rescaling, in the incompressible case, De Masi *et al.*[10] showed that for any given smooth solution $U(t, x)$ of the incompressible Navier–Stokes equation with density $\bar{\rho}$ and temperature $\bar{T}$ there exists a solution F^ε of (1.3) which differs by an infinitesimal of the order of ε^2 from the local Maxwellian M^ε of mean εU, constant density $\bar{\rho}$, and constant temperature $\bar{T}$. Thus, in both cases the system reaches a local equilibrium, characterized by the parameters (ρ, T, u) and $(\bar{\rho}, \bar{T}, \varepsilon U)$, respectively; physically the latter situation corresponds to looking at a macroscopic velocity field very small compared to the sound speed.

Here we use both the above results to study the limit of the Boltzmann processes corresponding to (1.1) and (1.3). Most of this paper is devoted to the more complex case of the parabolic rescaling. In this scaling the velocity is a fast mode while the position is a slow mode, so that the velocity equilibrates faster than the position. The jump frequency goes to infinity as ε^{-2} and the collisions give rise to a stochastic perturbation for the position of Brownian type. The choice for the velocity scaling brings up the Navier–Stokes velocity field as the drift of the limiting process. We prove that the position component of the Boltzmann process $(X^\varepsilon, V^\varepsilon)$ corresponding to (1.3) converges, in the sense of probability measures on the path space, to the solution of the following stochastic differential equation:

$$X(t) = X(0) + \int_0^t ds\, U(s, X(s)) + \sqrt{D}\, W(t) \tag{1.4}$$

where $U(t, x)$ is a solution of the incompressible Navier–Stokes equation and the diffusion coefficient is given by

$$D\delta_{ij} = 2 \int_0^\infty ds\, E[V_i(s)\, V_j(0)] \tag{1.5}$$

In (1.5) the expectation is taken with respect to the global equilibrium jump process. The diffusion coefficient has the usual form of a time average of a velocity–velocity time correlation.

The above result falls into the class of diffusion approximations for transport processes. Limit theorems of this type were proved in refs. 11 and 12 for Markov transport processes, with bounded velocities. Here we follow the approach proposed in Costantini[13], which works also in the case of non-compact state space (see also ref. 14 for yet another way to handle these problems). In our approach the process X^ε is represented by a stochastic equation driven by a suitable martingale. High-speed values are controlled by the moments of a function determined by the process and convergence is derived by the martingale central limit theorem together with a suitable stochastic averaging technique.

In the hyperbolic case, we show that the position component of the Boltzmann process $(x^\varepsilon, v^\varepsilon)$ corresponding to (1.1) converges, in the sense of probability measures on the path space, to the solution of the ordinary differential equation

$$\dot{x} = u(t, x) \tag{1.6}$$

where $u(t, x)$ is a solution of the compressible Euler equation. The proof is based on the same computations as for the parabolic case except that the martingale central limit theorem does not come into play.

From the physical point of view the limiting processes (1.4) and (1.6) represent the motion of a Boltzmann particle on the hydrodynamic time–space scales: in the hyperbolic case the particle moves deterministically following the Euler velocity field and the fluctuations of the microscopic velocities are noneffective. If one waits for a longer time, then, in the incompressible approximation, one can see the effect of the viscosity as a Brownian perturbation on the deterministic motion driven by the Navier–Stokes velocity field.

In the latter case an interesting question is the relation between the diffusion coefficient D of (1.4) and the viscosity which appears in the Navier–Stokes equation. This is a typical problem of the relation between self and bulk diffusion coefficients (see, for instance, ref. 15). In our case both are determined by the global equilibrium properties of the system and are related to relaxing times to the equilibrium. Since there exists a Lagrangian formulation of the Navier–Stokes equation[16,17] at macroscopic level which uses as stochastic characteristics the sample paths of a diffusion whose drift is the velocity field and whose diffusion coefficient (times the density) is the viscosity, it is natural to expect that this process coincides with our limiting process. The expression of the viscosity ν in terms of the Green–Kubo formula in the context of the Boltzmann linearized theory is[1,10]

$$\nu = 2\bar{T}^{-1} \int_0^\infty dt \int dv\, v_i v_j (e^{\mathscr{A}t} v_i v_j)\, M(v), \qquad i \neq j \tag{1.7}$$

where M is the global Maxwellian with density $\bar{\rho}$ and temperature $\bar{T}$ and $e^{\mathscr{A}t}$ is the semigroup corresponding to the linearized Boltzmann operator $\mathscr{A}$, which describes the evolution of a small perturbation of the hard-sphere Boltzmann system from the equilibrium. Thus, the right-hand side is the time average of a current–current correlation function at the equilibrium. Equation (1.7) is a kind of fluctuation-dissipation formula for the non-equilibrium case. On the other hand, the diffusion coefficient has a very similar expression (1.5), where, however, the operator involved is the adjoint $\bar{A}$ of the linear Boltzmann operator [given by (2.4) below with $f = M$]. From (1.5) and (1.7) the relation between D and ν appears to be connected to the spectral properties of the operators $\bar{A}$ and $\mathscr{A}$. A simpler case is the one of the Maxwellian molecules[1], for which the spectra of the operators $\bar{A}$ and $\mathscr{A}$ are almost explicitly known. Another possibility for obtaining some partial answers to the question is to look at some kinetic models with discrete velocities (cellular automata and some model Boltzmann equations[18]), for which the kinetic and hydrodynamic behaviors are known[18,19]. The advantage is that in this case the collision operator is a matrix; on the other hand, one has to deal with an intrinsec ambiguity in the definition of the linear operator (something like this happens for lattice gases[15]). At the moment we are studying the problem in these contexts.

2. FORMULATION OF THE PROBLEM AND RESULTS

Our first goal is to construct a stochastic process to describe the motion of the tracer particle in a Boltzmann fluid of hard spheres. Let $f(t, x, v)$, $t \in [0, t_0]$, $x \in \Pi$, $v \in R^3$, be the density of the fluid. The tracer particle moves with constant velocity; at random, conditionally exponential times it collides with the environment particles and its velocity jumps. The jump intensity $q(t, x, v)$ is given by

$$q(t, x, v) = \int_{(v_1 - v) \cdot \omega \geqslant 0} dv_1 \, d\omega (v_1 - v) \cdot \omega f(t, x, v_1) \tag{2.1}$$

where ω is a unit vector, and v and v_1 represent the velocities before the collision of the tracer particle and of the environment particle, respectively. At each collision a direction ω is chosen uniformly on the unit sphere and a particle of velocity v_1 is chosen with probability

$$\frac{1}{q(t, x, v)} [(v_1 - v) \cdot \omega]^{t} f(t, x, v_1) \, dv_1$$

Then the velocity of the tracer particle changes to a new velocity v' according to the rules of elastic collision

$$v' = v + [(v_1 - v) \cdot \omega] \omega \tag{2.2}$$

The dynamics just described corresponds to an infinitesimal generators of the form

$$L_t \gamma = v \cdot \nabla_x \gamma + A_{t,x} \gamma \tag{2.3}$$

$$A_{t,x} \gamma(t, x, v) = \int_{(v_1 - v) \cdot \omega \geqslant 0} dv_1 \, d\omega \times (v_1 - v) \cdot \omega f(t, x, v_1) \{\gamma(t, x, v') - \gamma(t, x, v)\} \tag{2.4}$$

In the sequel, whenever convenient, the indices t, x in $A_{t,x}$ will be dropped.

If we add the requirement that the tracer particle is identical to the environment particles, then its one-time probability distribution must have density proportional to $f(t, x, v)$. Therefore the forward Kolmogorov equation becomes the Boltzmann equation:

$$\partial_t f + v \cdot \nabla_x f = Q(f, f) \tag{2.5}$$

where Q is defined in (1.2) ($v_1' = v_1 - [(v_1 - v) \cdot \omega] \omega$).

Let $\mathscr{P}_{M^*} = \mathscr{P}_{M^*}(\Pi \times R^3)$ be the space of the probability measures P on $\Pi \times R^3$ such that

$$\int P(dv)(1 + |v|)(M^*)^{-1/2}(v) < \infty$$

where M^* is a global Maxwellian, endowed with the weak$*$ topology, and let $\mathscr{C}_{M^*}([s, t_0], \mathscr{P}_{M^*})$, $s \in [0, t_0)$, be the space of continuous functions $P_.$ from $[s, t_0]$ into $\mathscr{P}_{M^*}$ such that

$$\sup_{s \leqslant t \leqslant t_0} \int P_t(dv)(1 + |v|)(M^*)^{-1/2}(v) < \infty$$

Denote by $\mathscr{C}^{1,0}(\Pi \times R^3)$ the space of real-valued functions continuously differentiable in the first variable and continuous in the second one, and by $\mathscr{C}([s, t_0], \Pi)$ ($\mathscr{D}([s, t_0], R^3)$), $s \in [0, t_0)$, the space of continuous functions (right continuous functions with left-hand limits) from $[s, t_0]$ into Π (into R^3). For every solution f of the Boltzmann equation dominated by a Maxwellian M^* uniformly in t and x, the operator L_t defined by (2.3) uniquely determines a probability measure on $\mathscr{C}([0, t_0], \Pi) \times \mathscr{D}([0, t_0], R^3)$ with one-time density proportional to $f(t, \cdot, \cdot)$: In fact, for every $s \in [0, t_0)$, $P_0 \in \mathscr{P}_{M^*}$, it can be easily seen that there exists a solution to the local martingale problem for $(L_t, \mathscr{C}^{1,0}(\Pi \times R^3), (s, P_0))$; an *a priori* estimate based on ref. 20 shows that the function $P_.$ defined by the one-time distributions of any solution of the above local martingale problem belongs to $\mathscr{C}_{M^*}([s, t_0], \mathscr{P}_{M^*})$, and well-known techniques (see, for instance, ref. 21) yield that the (weak) forward Kolmogorov equation for L_t on $[s, t_0]$ with initial datum P_0 has at most one solution in this space. The corresponding canonical stochastic process (X, V) is a transport nonlinear Markov process driven by a jump process inhomogeneous in both time and space. It can also be represented by the following stochastic equations:

$$\begin{aligned} X(t) &= X(0) + \int_0^t ds\, V(s) \\ V(t) &= V(0) + \int_0^t ds\, Av(s, X(s), V(s)) + Y(t) \end{aligned} \tag{2.6}$$

where Y is a zero mean local square-integrable martingale (in fact a square-integrable martingale).

Nonlinear Markov processes and martingale problems associated with equations of Boltzmann type have been studied also in refs. 4, 5, 22, and 23.

Consider now the rescaled Boltzmann equation (1.2). Our starting point to obtain the diffusion approximation is the result of ref. 10, which is recalled next in a form tailored to our needs. Let $H_s(\Pi, R^k)$, $k \in N$, be the Sobolev space of order s.

Theorem 2.1 (De Masi, Esposito, Lebowitz). Let $U(t, x)$ be a solution of the incompressible Navier–Stokes equation on $[0, t_0]$ with density $\bar{\rho}$ and temperature $\bar{T}$, continuously differentiable in t, and such that $U(\cdot, 0) = U_0$, where U_0 is a divergenceless field in $H_s(\Pi, R^3)$, $s \geqslant 4$, and $U(t, \cdot) \in H_s(\Pi, R^3)$ for all $t \leqslant t_0$, and let M^ε be the local Maxwellian

$$M^\varepsilon(t, x, v) = \frac{\bar{\rho}}{(2\pi\bar{T})^{3/2}} \exp\left\{-\frac{|v - \varepsilon U(t, x)|^2}{2\bar{T}}\right\}$$

Then there is an $\varepsilon_0 > 0$ such that for every $\varepsilon \leqslant \varepsilon_0$ there exists, for a suitable initial datum $F^\varepsilon(0, \cdot, \cdot)$, a (strong) solution F^ε of (1.3) such that

$$\sup_{t,x} |F^\varepsilon(t, x, v) - M^\varepsilon(t, x, v)| \leqslant C_0^* \varepsilon^2 M^*(v), \qquad \forall v \in R^3 \tag{2.7}$$

for some positive constant C_0^*, for any global Maxwellian M^* with mean zero, density one, and temperature $T^* > 2\bar{T}$.

We consider the process $(X^\varepsilon, V^\varepsilon)$ corresponding to the solution F^ε of the rescaled Boltzmann equation (1.3) associated to the velocity field U by Theorem 2.1. $(X^\varepsilon, V^\varepsilon)$ satisfies the rescaled form of (2.6),

$$\begin{aligned} X^\varepsilon(t) &= X^\varepsilon(0) + \varepsilon^{-1} \int_0^t ds\, V^\varepsilon(s) \\ V^\varepsilon(t) &= V^\varepsilon(0) + \varepsilon^{-2} \int_0^t ds\, A^\varepsilon v(s, X^\varepsilon(s), V^\varepsilon(s)) + Y^\varepsilon(t) \end{aligned} \tag{2.8}$$

where A^ε is the operator defined by (2.4) with $f = F^\varepsilon$. In the sequel we will also use the operators $\bar{A}^\varepsilon$ and $\bar{A}$ defined by (2.4) with $f = M^\varepsilon$ and $f = M$, respectively, where M is the global Maxwellian with density $\bar{\rho}$, mean zero, and temperature $\bar{T}$. The corresponding jump intensities will be denoted by $\bar{q}^\varepsilon(t, x, v)$ and $\bar{q}(v)$. Our main result can be formulated as follows.

Theorem 2.2. The stochastic process X^ε converges, in the sense of probability measures on the path space, as ε goes to zero, to the solution of the stochastic differential equation

$$X(t) = X(0) + \int_0^t ds\, U(s, X(s)) + \sqrt{D}\, W_t$$

where $X(0)$ is uniformly distributed on Π and the diffusion coefficient is a constant given by

$$\delta_{ij} D = -\frac{2}{\bar{\rho}} \int dv\, M(v)\, v_i \bar{A}^{-1} v_j \tag{2.9}$$

where $\bar{A}^{-1} v$ is defined up to an additive constant.

The result of Lachowicz[9], as formulated in Theorem 2.3 below, allows us to prove convergence of the hyperbolically rescaled Boltzmann process (Theorem 2.4).

Theorem 2.3 (Lachowicz). Let (ρ, T, u) be a smooth solution of the compressible Euler equation on $[0, t_0]$, such that

$$\inf_{t,x} \rho(t, x) = c_\rho > 0, \qquad \inf_{t,x} T(t, x) = c_T > 0$$

and let m be the local Maxwellian with parameters (ρ, T, u),

$$m(t, x, v) = \frac{\rho(t, x)}{[2\pi T(t, x)]^{3/2}} \exp\left\{ -\frac{|v - u(t, x)|^2}{2T(t, x)} \right\}$$

Then there is an $\varepsilon_0 > 0$ such that for every $\varepsilon \leqslant \varepsilon_0$ there exists, for a suitable initial datum $f^\varepsilon(0, \cdot, \cdot)$, a (strong) solution f^ε of (1.1) such that

$$\sup_{t,x} |f^\varepsilon(t, x, v) - m(t, x, v)| \leqslant c_0^* \varepsilon M^*(v), \qquad \forall v \in R^3 \tag{2.10}$$

for some positive constant c_0^*, for any global Maxwellian M^* with mean zero, density one, and temperature $T^* > 2 \sup_{t,x} T(t, x)$.

Let $(x^\varepsilon, v^\varepsilon)$ be the process corresponding to the solution f^ε of (1.1) associated to (ρ, T, u) by Theorem 2.3. $(x^\varepsilon, v^\varepsilon)$ satisfies

$$\begin{aligned} x^\varepsilon(t) &= x^\varepsilon(0) + \int_0^t ds\, v^\varepsilon(s) \\ v^\varepsilon(t) &= v^\varepsilon(0) + \varepsilon^{-1} \int_0^t ds\, a^\varepsilon v^\varepsilon(s, x^\varepsilon(s), v^\varepsilon(s)) + y^\varepsilon(t) \end{aligned} \tag{2.11}$$

where a^ε is the operator defined by (2.4) with $f = f^\varepsilon$ and $y^\varepsilon(t)$ is a zero mean local square-integrable martingale (in fact a square-integrable martingale).

Theorem 2.4. The stochastic process x^ε converges, in the sense of probability measures on the path space, as ε goes to zero, to the solution of the ordinary differential equation

$$\dot{x} = u(t, x)$$

with initial condition $x(0)$ distributed on Π with probability density proportional to $\rho(0, \cdot)$. ∎

3. DISCUSSION OF THE MATHEMATICAL TECHNIQUES AND PROOFS

As anticipated in the Introduction, we will prove Theorem 2.2 (limit under parabolic scaling) first. In order to illustrate our approach to the diffusion approximation of transport processes, let us recall the standard approach as presented, for instance, in ref. 11. In the latter setup one considers the collection of the solutions ψ^ε of the backward Kolmogorov equation for $(X^\varepsilon(t), V^\varepsilon(t))$,

$$\psi^\varepsilon(t, x, v) = E_{t,x,v}[g(X^\varepsilon(s), V^\varepsilon(s))], \qquad t \leqslant s \tag{3.1}$$

as g varies in a suitable class of bounded continuous functions. ψ^ε is developed in a formal Taylor expansion in powers of ε, with coefficients ψ_k, which is substituted in the backward Kolmogorov equation. In our case an additional difficulty arises from the fact that the operator A^ε itself depends on ε by F^ε. However, since the operator depends linearly on F^ε, we know by Theorem 2.1 that

$$A^\varepsilon = \bar{A} + O(\varepsilon) \tag{3.2}$$

By taking into account (3.2), a sequence of terms appears in the backward Kolmogorov equation, each multiplied by a power of ε starting with the order ε^{-2}. The terms multiplied by the divergent powers of ε have to be set equal to zero:

$$\varepsilon^{-2}: \quad \bar{A}\psi_0 = 0 \tag{3.3}$$

$$\varepsilon^{-1}: \quad \bar{A}\psi_1 + v \cdot \nabla_x \psi_0 = 0 \tag{3.4}$$

By equating terms of the same order in ε starting from the order zero, one obtains a sequence of equations for the remaining coefficients of the Taylor expansion of ψ^ε. If one can prove the validity of the formal Taylor expansion (usually by analytic tools), this determines the form of the infinitesimal generator of the limiting process. Convergence can then be proved by Markov processes techniques, such as martingale problem techniques.

In contrast, our approach consists in working directly on the sample paths of the process X^ε. We consider only the function $g(x, v) = x$ (in the sequel all operators are extended in the obvious manner to vector-valued functions). With this choice of g we have $\psi_0 = x$ and

$$\bar{A}\psi_1 = -v \tag{3.5}$$

Existence and regularity of solutions to (3.5) are ensured by the following Lemma 3.1. Let $L^2(E, \mu; R^k)$, $k \in N$, be the space of R^k-valued, square-integrable functions on a finite measure space (E, μ); for $k = 1$ we will write simply $L^2(E, \mu)$. Also, for any function $g \in C^1(R^3, R^k)$, $k \in N$, let ∂g denote the Jacobian matrix; $|\cdot|$ will denote indifferently the modulus of a vector or the norm of a matrix.

Lemma 3.1. Equation (3.5) has a solution in $L^2(R^3, M\bar{q}\, dv; R^3)$ and the solution is unique up to an additive constant. For any solution ϕ of (3.5), for every $\alpha > 0$,

$$\sup_v M^{1/\alpha}(v)\, |\phi(v)| < \infty \tag{3.6}$$

Moreover, $\phi \in C^1(R^3, R^3)$ and, for every $\alpha > 0$,

$$\sup_v M^{1/\alpha}(v)\, |\partial\phi(v)| < \infty \tag{3.7}$$

Proof. See the Appendix. ▮

To fix a specific solution of (3.5), in the sequel we will denote by ϕ the solution of (3.5) such that

$$\int dv\, M(v)\, \bar{q}(v)\, \phi(v) = 0 \tag{3.8}$$

By setting

$$\phi(v - \varepsilon U(t, x)) = \phi^\varepsilon(t, x, v) \tag{3.9}$$

(3.5) takes the form

$$\bar{A}^\varepsilon_{t,x}\phi^\varepsilon(t, x, v) = -(v - \varepsilon U(t, x)) \tag{3.10}$$

Let

$$R^\varepsilon(t, x, v) = \varepsilon^{-2}[A^\varepsilon_{t,x}\phi^\varepsilon(t, x, v) - \bar{A}^\varepsilon_{t,x}\phi^\varepsilon(t, x, v)] \tag{3.11}$$

By applying Ito's formula to ϕ^ε, we get

$$\begin{aligned}\phi^\varepsilon(t, X^\varepsilon(t), V^\varepsilon(t)) - \phi^\varepsilon(0, X^\varepsilon(0), V^\varepsilon(0))\\ = \int_0^t ds\, \partial_s \phi^\varepsilon(s, X^\varepsilon(s), V^\varepsilon(s)) + \varepsilon^{-1} \int_0^t ds\, (\partial_x \phi^\varepsilon v)(s, X^\varepsilon(s), V^\varepsilon(s))\\ + \varepsilon^{-2} \int_0^t ds\, A^\varepsilon \phi^\varepsilon(s, X^\varepsilon(s), V^\varepsilon(s)) + Z^\varepsilon(t)\end{aligned} \tag{3.12}$$

where

$$\partial_x g\, v = \sum_j \frac{\partial g_i}{\partial x_j} v_j$$

and $Z^\varepsilon(t)$ is a zero mean local square-integrable martingale [in fact a square-integrable martingale by Theorem 2.1, (3.15)–(3.17) below] with

$$\begin{aligned}\langle Z^\varepsilon \rangle(t) = \varepsilon^{-2} \int_0^t ds\, A^\varepsilon(\phi^\varepsilon \phi^{\varepsilon *})(s, X^\varepsilon(s), V^\varepsilon(s))\\ - \varepsilon^{-2} \int_0^t ds\, (A^\varepsilon \phi^\varepsilon) \phi^{\varepsilon *}(s, X^\varepsilon(s), V^\varepsilon(s))\\ - \varepsilon^{-2} \int_0^t ds\, \phi^\varepsilon (A^\varepsilon \phi^\varepsilon)^* (s, X^\varepsilon(s), V^\varepsilon(s))\end{aligned} \tag{3.13}$$

where * denotes transposition (cf. ref. 24, Chapter 2, Section 6). By combining (3.10)–(3.12), solving for $\int_0^t ds\, V^\varepsilon(s)$, and substituting into (2.8), we are able to obtain the following expression for the position component of the process:

$$\begin{aligned}X^\varepsilon(t) = X^\varepsilon(0) + \int_0^t ds\, U(s, X^\varepsilon(s)) + \varepsilon Z^\varepsilon(t) - \varepsilon[\phi^\varepsilon(t, X^\varepsilon(t), V^\varepsilon(t))\\ - \phi^\varepsilon(0, X^\varepsilon(0), V^\varepsilon(0))] + \varepsilon \int_0^t ds\, \partial_s \phi^\varepsilon(s, X^\varepsilon(s), V^\varepsilon(s))\\ + \int_0^t ds\, (\partial_x \phi^\varepsilon v)(s, X^\varepsilon(s), V^\varepsilon(s)) + \varepsilon \int_0^t ds\, R^\varepsilon(s, X^\varepsilon(s), V^\varepsilon(s))\end{aligned} \tag{3.14}$$

Note that by (3.9) also the term $\partial_x \phi^\varepsilon$ is of order ε.

We now proceed to analyze each summand in the right-hand side of (3.14). Particular care is necessary to deal with the fourth summand, which is treated in Lemma 3.2. The martingale term $\varepsilon Z^\varepsilon$ is controlled by $\langle \varepsilon Z^\varepsilon \rangle = \varepsilon^2 \langle Z^\varepsilon \rangle$, which in its turn is handled by proving a suitable law of large numbers (Lemma 3.5).

Clearly, by Theorem 2.1 one can suppose that

$$\inf_{\varepsilon \leqslant \varepsilon_0} \bar{\rho}^\varepsilon > 0 \tag{3.15}$$

where, as in the Introduction,

$$\bar{\rho}^\varepsilon = \int dx\, dv\, F^\varepsilon(t, x, v)$$

and choose C_0^* such that

$$\sup_{\varepsilon \leqslant \varepsilon_0} \sup_{t, x} M^\varepsilon(t, x, v) \leqslant C_0^* M^*(v), \qquad \forall v \in R^3 \tag{3.16}$$

Moreover, by Lemma 3.1 and (3.9), for any temperature T^* and for every $\alpha > 0$, there exists $C_\alpha^* > 0$ such that for all $\varepsilon \leqslant \varepsilon_0$, $t \leqslant t_0$, $v \in R^3$,

$$\begin{aligned} |\phi^\varepsilon(t, x, v)| &\leqslant C_\alpha^* M^*(v)^{-1/\alpha} \\ |\partial\phi(v - \varepsilon U(t, x))| &\leqslant C_\alpha^* M^*(v)^{-1/\alpha} \\ |\phi^\varepsilon(t, x, v')| &\leqslant C_\alpha^* M^*(v)^{-1/\alpha} M^*(v_1)^{-1/\alpha} \end{aligned} \tag{3.17}$$

where v' is defined in (2.2).

Lemma 3.2. For any positive δ

$$\sup_{\varepsilon \leqslant \varepsilon_0} E[\sup_{t \leqslant t_0} \varepsilon^2 |\phi^\varepsilon(t, X^\varepsilon(t), V^\varepsilon(t))|^{2(1+\delta)}] < +\infty \tag{3.18}$$

Proof. By applying Ito's formula to $|\phi^\varepsilon(t, x, v)|^{2(1+\delta)}$, we get

$$\begin{aligned} &|\phi^\varepsilon(t, X^\varepsilon(t), V^\varepsilon(t))|^{2(1+\delta)} \\ &\quad = |\phi^\varepsilon(0, X^\varepsilon(0), V^\varepsilon(0))|^{2(1+\delta)} + \int_0^t ds\, \partial_s |\phi^\varepsilon|^{2(1+\delta)}(s, X^\varepsilon(s), V^\varepsilon(s)) \\ &\qquad + \varepsilon^{-1} \int_0^t ds\, (\partial_x |\phi^\varepsilon|^{2(1+\delta)} v)(s, X^\varepsilon(s), V^\varepsilon(s)) \\ &\qquad + \varepsilon^{-2} \int_0^t ds\, (A^\varepsilon |\phi^\varepsilon|^{2(1+\delta)})(s, X^\varepsilon(s), V^\varepsilon(s)) + Z_\delta^\varepsilon(t) \end{aligned}$$

where Z_δ^ε is a zero mean local square-integrable martingale and

$$\begin{aligned} \langle Z_\delta^\varepsilon \rangle(t) = \varepsilon^{-2} &\int_0^t ds\, (A^\varepsilon |\phi^\varepsilon|^{4(1+\delta)})(s, X^\varepsilon(s), V^\varepsilon(s)) \\ &- 2\varepsilon^{-2} \int_0^t ds\, [(A^\varepsilon |\phi^\varepsilon|^{2(1+\delta)}) |\phi^\varepsilon|^{2(1+\delta)}](s, X^\varepsilon(s), V^\varepsilon(s)) \end{aligned}$$

The expected value of $\langle Z^{\varepsilon}_{\delta}\rangle(s)$ is bounded by a constant times

$$\varepsilon^{-2}\frac{1}{\bar{\rho}^{\varepsilon}}\left\{\int dx\, dv \int dv_1\, |v_1 - v|\; |\phi^{\varepsilon}|^{4(1+\delta)}|\,(s, x, v')\, F^{\varepsilon}(s, x, v_1)\, F^{\varepsilon}(s, x, v)\right.$$

$$\left.+\int dx\, dv\, q^{\varepsilon}(s, x, v)(|\phi^{\varepsilon}|^{4(1+\delta)}(s, x, v) + 1)\, F^{\varepsilon}(s, x, v)\right\}$$

which is finite by Theorem 2.1, (3.15)–(3.17). Therefore Z^{ε}_{δ} is a square-integrable martingale and we have, by Doob's inequality,

$$\begin{aligned}
&E[\sup_{t\leqslant t_0} \varepsilon^2 |\phi^{\varepsilon}(t, X^{\varepsilon}(t), V^{\varepsilon}(t))|^{2(1+\delta)}]\\
&\quad\leqslant \varepsilon^2 E[|\phi^{\varepsilon}(0, X^{\varepsilon}(0), V^{\varepsilon}(0))|^{2(1+\delta)}]\\
&\qquad+\varepsilon^2 t_0 \sup_{s\leqslant t_0} E[|\partial_s |\phi^{\varepsilon}|^{2(1+\delta)}(s, X^{\varepsilon}(s), V^{\varepsilon}(s))|]\\
&\qquad+\varepsilon t_0 \sup_{s\leqslant t_0} E[|(\partial |\phi^{\varepsilon}|^{2(1+\delta)} v)(s, X^{\varepsilon}(s), V^{\varepsilon}(s))|]\\
&\qquad+t_0 \sup_{s\leqslant t_0} E[|(A^{\varepsilon} |\phi^{\varepsilon}|^{2(1+\delta)})(s, X^{\varepsilon}(s), V^{\varepsilon}(s))|]\\
&\qquad+2\varepsilon\{t_0 \sup_{s\leqslant t_0} E[(A^{\varepsilon} |\phi^{\varepsilon}|^{4(1+\delta)})(s, X^{\varepsilon}(s), V^{\varepsilon}(s))\\
&\qquad-2[(A^{\varepsilon} |\phi^{\varepsilon}|^{2(1+\delta)})\, |\phi^{\varepsilon}|^{2(1+\delta)}](s, X^{\varepsilon}(s), V^{\varepsilon}(s))]\}^{1/2}
\end{aligned}$$

where all the summands on the right-hand side are uniformly bounded in ε by Theorem 2.1, (3.15)–(3.17). ▌

Let

$$\begin{aligned}
U^{\varepsilon}(t) = &-\varepsilon[\phi^{\varepsilon}(t, X^{\varepsilon}(t), V^{\varepsilon}(t)) - \phi^{\varepsilon}(0, X^{\varepsilon}(0), V^{\varepsilon}(0))]\\
&+\varepsilon\int_0^t ds\, \partial_s \phi^{\varepsilon}(s, X^{\varepsilon}(s), V^{\varepsilon}(s))\\
&+\int_0^t ds\, (\partial_x \phi^{\varepsilon}\, v)(s, X^{\varepsilon}(s), V^{\varepsilon}(s))\\
&+\varepsilon\int_0^t ds\, R^{\varepsilon}(s, X^{\varepsilon}(s), V^{\varepsilon}(s))
\end{aligned}\tag{3.19}$$

Lemma 3.3. The process U^{ε} converges to zero, in the sense of probability measures on the path space, as ε goes to zero.

Proof. We have, for any $\delta > 0$,

$$
\begin{aligned}
&E[\sup_{t \leqslant t_0} |U^\varepsilon(t)|] \\
&\quad \leqslant 2\varepsilon^{1-1/(1+\delta)}\{E[\sup_{t \leqslant t_0} \varepsilon^2 |\phi^\varepsilon(t, X^\varepsilon(t), V^\varepsilon(t))|^{2(1+\delta)}]\}^{1/[2(1+\delta)]} \\
&\qquad + \varepsilon t_0 \sup_{s \leqslant t_0} E[|R^\varepsilon(s, X^\varepsilon(s), V^\varepsilon(s))|] \\
&\qquad\quad + \varepsilon t_0 \sup_{s \leqslant t_0} E[|\partial_s \phi^\varepsilon(s, X^\varepsilon(s), V^\varepsilon(s))|] \\
&\qquad\quad + \varepsilon t_0 \sup_{s \leqslant t_0} E[|\partial\phi(V^\varepsilon(s) - \varepsilon U(s, X^\varepsilon(s)))\, \partial_x U(s, X^\varepsilon(s))\, V^\varepsilon(s)|]
\end{aligned}
$$

The first summand is dominated by a constant times $2\varepsilon^{1-1/(1+\delta)}$ by Lemma 3.2. As far as the second term is concerned, we have

$$
\begin{aligned}
&\sup_{s \leqslant t_0} E[|R^\varepsilon(s, X^\varepsilon(s), V^\varepsilon(s))|] \\
&\quad = \varepsilon^{-2} \sup_{s \leqslant t_0} E[|A^\varepsilon\phi^\varepsilon(s, X^\varepsilon(s), V^\varepsilon(s)) - \bar{A}^\varepsilon\phi^\varepsilon(s, X^\varepsilon(s), V^\varepsilon(s))|] \\
&\quad \leqslant \varepsilon^{-2} \frac{4\pi}{\bar{\rho}^\varepsilon} \sup_{s \leqslant t_0} \left\{ \int dx\, dv \int dv_1 \right. \\
&\qquad \times |v_1 - v|\, |\phi^\varepsilon(s, x, v') - \phi^\varepsilon(s, x, v)|\, |F^\varepsilon(s, x, v_1) - M^\varepsilon(s, x, v_1)| \\
&\qquad \times M^\varepsilon(s, x, v) + \int dx\, dv \int dv_1\, |v_1 - v|\, |\phi^\varepsilon(s, x, v') - \phi^\varepsilon(s, x, v)| \\
&\qquad \left. \times |F^\varepsilon(s, x, v_1) - M^\varepsilon(s, x, v_1)|\, |F^\varepsilon(s, x, v) - M^\varepsilon(s, x, v)| \right\}
\end{aligned}
$$

where the right-hand side is bounded uniformly in ε by Theorem 2.1, (3.15)–(3.17). Analogously, the last two summands are dominated by εt_0 times a constant, by Theorem 2.1, (3.15)–(3.17). ∎

Lemma 3.4. For every sequence $\{\varepsilon_k\}$ converging to zero, the family of stochastic processes $\{X^{\varepsilon_k}\}$ is relatively compact.

Proof. By (3.14)

$$X^{\varepsilon_k}(t) = X^{\varepsilon_k}(0) + \int_0^t ds\, U(s, X^{\varepsilon_k}(s)) + U^{\varepsilon_k}(t) + \varepsilon_k Z^{\varepsilon_k}(t)$$

The sequence of the sums of the first three terms is relatively compact by Theorem 2.1, Lemma 3.3., and the fact that $\sup_{s,x} |U(s, x)| < +\infty$. Set

$$\zeta^\varepsilon(s) = [A^\varepsilon |\phi^\varepsilon|^2 - 2\phi^\varepsilon \cdot (A^\varepsilon\phi^\varepsilon)](s, X^\varepsilon(s), V^\varepsilon(s))$$

Then, we have, by (3.13),

$$\operatorname{tr} E[\langle Z^\varepsilon\rangle(t)] = \int_0^t ds\, E[\zeta^\varepsilon(s)]$$

$$E[\zeta^\varepsilon(s)] = \frac{1}{\bar\rho^\varepsilon} \int dx\, dv\, F^\varepsilon(s, x, v) \int_{(v_1 - v)\cdot\omega \geqslant 0} dv_1\, d\omega \\ \times (v_1 - v)\cdot\omega\, |\phi^\varepsilon(s, x, v') - \phi^\varepsilon(s, x, v)|^2\, F^\varepsilon(s, x, v_1)$$

which is finite by Theorem 2.1, (3.15)–(3.17), so that Z^ε is a square-integrable martingale and, denoting by $\mathscr{F}_t^\varepsilon$ the σ-algebra generated by $\varepsilon Z^\varepsilon(s)$, $s \leqslant t$, for any $\eta > 0$, $0 \leqslant t \leqslant t + \eta \leqslant t_0$,

$$E[|\varepsilon Z^\varepsilon(t+\eta) - \varepsilon Z^\varepsilon(t)|^2 \mid \mathscr{F}_t^\varepsilon] \\ = \operatorname{tr} E[\langle \varepsilon Z^\varepsilon\rangle(t+\eta) - \langle \varepsilon Z^\varepsilon\rangle(t) \mid \mathscr{F}_t^\varepsilon] \\ \leqslant E\left[\sup_{t \leqslant t_0 - \eta} \int_t^{t+\eta} ds\, \zeta^\varepsilon(s) \mid \mathscr{F}_t^\varepsilon\right]$$

Since, for any $\delta > 0$,

$$E\left[\sup_{t \leqslant t_0 - \eta} \int_t^{t+\eta} ds\, \zeta^\varepsilon(s)\right] \\ \leqslant \eta^{1 - 1/(1+\delta)}\, t_0^{1/(1+\delta)} \{\sup_{s \leqslant t_0} E[\zeta^\varepsilon(s)^{1+\delta}]\}^{1/(1+\delta)}$$

where the expectation in the last factor of the right-hand side is bounded, uniformly in ε, by Theorem 2.1, (3.15)–(3.17), it follows that $\{\varepsilon_k Z^{\varepsilon_k}\}$ is relatively compact by ref. 24, Theorem 8.6, Chapter 3. Finally, $\{X^{\varepsilon_k}\}$ is relatively compact because all limit points of $X^{\varepsilon_k}(0) + \int_0^{\cdot} ds\, U(s, X^{\varepsilon_k}(s)) + U^{\varepsilon_k}(\cdot)$ are continuous. ∎

For every measurable function g such that

$$\int dv\, M(v)\, |g(t, x, v)| < \infty, \qquad \forall (t, x) \tag{3.20}$$

set

$$\bar g(t, x) = \frac{1}{\bar\rho} \int dv\, M(v)\, g(t, x, v) \tag{3.21}$$

Lemma 3.5. Let g be a real-valued measurable function such that, for some $\alpha > 1$,

$$\int_0^{t_0} dt \int dx\, dv\, \bar q(v)^{-1}\, M(v)^{1/\alpha}\, |g(t, x, v)|^2 < \infty \tag{3.22}$$

where $\bar{q}(v)$ is defined by (2.1) with $f=M$. Then

$$\lim_{\varepsilon\to 0} E\left[\sup_{t\leqslant t_0}\left|\int_0^t ds\,[g(s, X^\varepsilon(s), V^\varepsilon(s))-\bar{g}(s, X^\varepsilon(s))]\right|\right]=0 \tag{3.23}$$

Proof. It is proved below in Theorem A.1 that for every $\alpha>1$ $\bar{A}$ maps continuously $L^2([0, t_0]\times\Pi\times R^3, \bar{q}M^{1/\alpha}\,ds\,dx\,dv)$ onto the subspace of $L^2([0, t_0]\times\Pi\times R^3, \bar{q}^{-1}M^{1/\alpha}\,ds\,dx\,dv)$ of the functions g such that $\bar{g}(t, x)=0$, which we call $\mathscr{R}_\alpha(\bar{A})$. Thus, the linear span of the functions $\gamma_1(t, x)\,\bar{A}\gamma_2(v)$ with γ_1 continuously differentiable and γ_2 bounded and continuous is dense in $\mathscr{R}_\alpha(\bar{A})$; moreover for $g, h\in\mathscr{R}_\alpha(\bar{A})$, by choosing $T^*<2\alpha\bar{T}$ in Theorem 2.1, one can see that

$$\int_0^{t_0} ds\,E[|g(s, X^\varepsilon(s), V^\varepsilon(s))-R(s, X^\varepsilon(s), V^\varepsilon(s))|]$$

is bounded, uniformly in ε, by a constant times

$$\left\{\int_0^{t_0} ds\int dx\,dv\,|g(s, x, v)-h(s, x, v)|^2 M^{1/\alpha}(v)\,\bar{q}^{-1}(v)\right\}^{1/2}$$

On the other hand, Theorem 2.1 implies that

$$\lim_{\varepsilon\to 0}\int_0^{t_0} ds\,E[|\gamma_1(s, X^\varepsilon(s))\,\bar{A}\gamma_2(V^\varepsilon(s)) \\ -\gamma_1(s, X^\varepsilon(s))\,A^\varepsilon\gamma_2(s, X^\varepsilon(s), V^\varepsilon(s))|]=0$$

Therefore it is enough to prove that

$$\lim_{\varepsilon\to 0} E\left[\sup_{t\leqslant t_0}\left|\int_0^t ds\,A^\varepsilon\gamma(s, X^\varepsilon(s), V^\varepsilon(s))\right|\right]=0 \tag{3.24}$$

for any function γ bounded, continuous in v, and continuously differentiable in t and x, with bounded derivatives. By applying Ito's formula to γ, we get

$$-\int_0^t ds\,A^\varepsilon\gamma(s, X^\varepsilon(s), V^\varepsilon(s)) \\ =\varepsilon^2[\gamma(0, X^\varepsilon(0), V^\varepsilon(0))-\gamma(t, X^\varepsilon(t), V^\varepsilon(t))] \\ +\varepsilon^2\int_0^t ds\,\partial_s\gamma(s, X^\varepsilon(s), V^\varepsilon(s))+\varepsilon\int_0^t ds\,(\partial_x\gamma\,v)(s, X^\varepsilon(s), V^\varepsilon(s))+\varepsilon^2 Z^\varepsilon_\gamma(t)$$

where Z_γ^ε is a mean-zero square-integrable martingale with

$$\langle Z_\gamma^\varepsilon\rangle(t) = \varepsilon^{-2}\int_0^t ds\,(A^\varepsilon|\gamma|^2)(s, X^\varepsilon(s), V^\varepsilon(s))$$

$$-2\varepsilon^{-2}\int_0^t ds\,(\gamma A^\varepsilon\gamma)(s, X^\varepsilon(s), V^\varepsilon(s))$$

Hence, by Doob's inequality we have

$$E\left[\sup_{t\leqslant t_0}\left|\int_0^t ds\,A^\varepsilon\gamma(s, X^\varepsilon(s), V^\varepsilon(s))\right|\right]$$

$$\leqslant 2\varepsilon^2\|\gamma\|_\infty + \varepsilon^2 t_0\|\partial_s\gamma\|_\infty + \varepsilon t_0\|\partial_x\gamma\|_\infty \sup_{\varepsilon\leqslant\varepsilon_0}\sup_{t\leqslant t_0} E[|V^\varepsilon(t)|]$$

$$+2\varepsilon\{t_0\sup_{s\leqslant t_0} E[(A^\varepsilon|\gamma|^2)(s, X^\varepsilon(s), V^\varepsilon(s)) - 2(\gamma A^\varepsilon\gamma)(s, X^\varepsilon(s), V^\varepsilon(s))]\}^{1/2}$$

where the right-hand side vanishes as ε goes to zero. ∎

Proof of Theorem 2.2. For every sequence $\{\varepsilon_k\}$ converging to zero, we have

$$E[\sup_{t\leqslant t_0}|\langle\varepsilon_k Z^{\varepsilon_k}\rangle(t) - tDI|]$$

$$\leqslant t_0\sup_{s\leqslant t_0} E[|\{A^\varepsilon(\phi^\varepsilon\phi^{\varepsilon *}) - (A^\varepsilon\phi^\varepsilon)\phi^{\varepsilon *} - \phi^\varepsilon(A^\varepsilon\phi^\varepsilon)^*$$

$$-\bar{A}(\phi\phi^*) + (\bar{A}\phi)\phi^* + \phi(\bar{A}\phi)^*\}(s, X^\varepsilon(s), V^\varepsilon(s))|]$$

$$+E\left[\sup_{t\leqslant t_0}\left|\int_0^t ds\,\{\bar{A}(\phi\phi^*) - (\bar{A}\phi)\phi^* - \phi(\bar{A}\phi)^* - DI\}\,(V^\varepsilon(s))\right|\right]$$

where D is defined by (2.9). The first term on the right-hand side converges to zero by Theorem 2.1, Lemma 3.1, (3.15)–(3.17) (see proofs of Lemmas 3.2 and 3.4), while the second term converges to zero by Lemmas 3.1 and 3.5. In addition,

$$E[\sup_{t\leqslant t_0}|\varepsilon_k Z^{\varepsilon_k}(t) - \varepsilon_k Z^{\varepsilon_k}(t^-)|^2]$$

$$= E[\sup_{t\leqslant t_0}\varepsilon_k^2|\phi^{\varepsilon_k}(t, X^{\varepsilon_k}(t), V^{\varepsilon_k}(t)) - \phi^{\varepsilon_k}(t, X^{\varepsilon_k}(t), V^{\varepsilon_k}(t^-))|^2]$$

and the right-hand side goes to zero, as $k\to\infty$, by Lemma 3.2. Therefore by the martingale central limit theorem (ref. 24, Chapter 7, Theorem 1.4), $\{\varepsilon_k Z^{\varepsilon_k}\}$ converges in the sense of probability measures on the path space to $\sqrt{D}\,W$, where W is a standard Brownian motion. By Lemma 3.3, all limit points of $\{X^{\varepsilon_k}\}$ are solutions of the stochastic differential equation (1.4). As (1.4) has a unique solution (see, for instance, ref. 24, Theorem 3.7, Chapter 5), the assertion is proved. ▌

We now turn to the hyperbolic case.

Proof of Theorem 2.4. The function

$$\varphi(t, x, v) = \frac{\bar{\rho}}{\rho(t, x)} \phi\left(\left(\frac{\bar{T}}{T(t, x)}\right)^{1/2} [v - u(t, x)]\right) \tag{3.25}$$

ϕ being defined by Lemma 3.1 and (3.8), satisfies

$$a\varphi(t, x, v) = -(v - u(t, x)) \tag{3.26}$$

where a is the operator defined by (2.4) with $f = m$ and m is the local Maxwellian of Theorem 2.3. Since u is smooth, by Lemma 3.1, φ belongs to $\mathscr{C}^1([0, t_0] \times \Pi \times R^3)$ and, for every $\alpha > 0$, there exist constants c_α^* such that

$$\begin{aligned}
\sup_{t,x} |\varphi(t, x, v)| &\leqslant c_\alpha^* M^*(v)^{-1/\alpha} \\
\sup_{t,x} |\partial_t \varphi(t, x, v)| &\leqslant c_\alpha^* M^*(v)^{-1/\alpha} \\
\sup_{t,x} |\partial_x \varphi(t, x, v) v| &\leqslant c_\alpha^* M^*(v)^{-1/\alpha}
\end{aligned} \tag{3.27}$$

where M^* is as in Theorem 2.3. By applying Ito's formula to φ, we get, setting

$$r^\varepsilon(t, x, v) = \varepsilon^{-1}[a^\varepsilon \varphi - a\varphi](t, x, v)$$

$$\begin{aligned}
x^\varepsilon(t) = x^\varepsilon(0) &+ \int_0^t ds\, u(s, x^\varepsilon(s)) - \varepsilon[\varphi(t, x^\varepsilon(t), v^\varepsilon(t)) \\
&- \varphi(0, x^\varepsilon(0), v^\varepsilon(0))] + \varepsilon \int_0^t ds\, \partial_s \varphi^\varepsilon(s, x^\varepsilon(s), v^s(s)) \\
&+ \varepsilon \int_0^t ds\, \partial_x \varphi(s, x^\varepsilon(s), v^\varepsilon(s))\, v^\varepsilon(s) \\
&+ \varepsilon \int_0^t ds\, r^\varepsilon(s, x^\varepsilon(s), v^\varepsilon(s)) + \varepsilon z^\varepsilon(t)
\end{aligned} \tag{3.28}$$

where z^ε is a zero-mean square-integrable martingale with

$$\langle z^\varepsilon \rangle(t) = \varepsilon^{-1} \int_0^t ds\, [a^\varepsilon(\varphi\varphi^*) - (a^\varepsilon\varphi)\varphi^* - \varphi(a^\varepsilon\varphi)^*](s, x^\varepsilon(s), v^\varepsilon(s)) \tag{3.29}$$

Then the same computations as in Lemmas 3.2 and 3.3 show that

$$\lim_{\varepsilon \to 0} E[\sup_{t \leqslant t_0} \varepsilon \,|\, \varphi(0, x^\varepsilon(0), v^\varepsilon(0)) - \varphi(t, x^\varepsilon(t), v^\varepsilon(t))|] = 0$$

and

$$\lim_{\varepsilon \to 0} E\left[\sup_{t \leqslant t_0} \varepsilon \left| \int_0^t ds\, [\partial_s \varphi + \partial_x \varphi v + r^\varepsilon](s, x^\varepsilon(s), v^\varepsilon(s)) \right| \right] = 0$$

In addition,

$$\begin{aligned} &\sup_{s \leqslant t_0} E[\operatorname{tr}[a^\varepsilon(\varphi\varphi^*) - (a^\varepsilon\varphi)\varphi^* - \varphi(a^\varepsilon\varphi^*)](s, x^\varepsilon(s), v^\varepsilon(s))] \\ &\quad = \sup_{s \leqslant t_0} \frac{1}{\bar{\rho}^\varepsilon} \int dx\, dv \int_{(v_1 - v) \cdot \omega \geqslant 0} d\omega\, dv_1 \\ &\qquad \times (v_1 - v) \cdot \omega \,|\varphi(s, x, v) - \varphi(s, x, v')|^2 f^\varepsilon(s, x, v_1) f^\varepsilon(s, x, v) \end{aligned}$$

[$\bar{\rho}^\varepsilon = \int dx\, dv\, f^\varepsilon(s, x, v)$], where the right-hand side is bounded uniformly in ε by Theorem 2.3 and (3.27). Since, by Doob's inequality,

$$E[\sup_{t \leqslant t_0} |\varepsilon z^\varepsilon(t)|^2] \leqslant 4\varepsilon^2 \operatorname{tr} E[\langle z^\varepsilon \rangle(t_0)]$$

$\varepsilon z^\varepsilon$ converges to zero as ε goes to zero. Therefore, for every sequence $\{\varepsilon_k\}$ converging to zero, $\{x^{\varepsilon_k}\}$ is relatively compact and all its limit points satisfy (1.5); as (1.5) has a unique solution, this proves the assertion. ∎

APPENDIX

Theorem 3.1 is a consequence of the following result.

Theorem A.1. For every $g \in L^2(R^3, M\bar{q}^{-1}\, dv)$ such that

$$\int dv\, g(v)\, M(v) = 0 \tag{A.1}$$

the equation

$$\bar{A}\gamma = g \tag{A.2}$$

has a solution in $L^2(R^3, M\bar{q}\, dv)$, unique up to an additive constant. Moreover, for every $\alpha \geqslant 1$, if $g \in L^2(R^3, M^{1/\alpha}\bar{q}^{-1}\, dv)$, then any solution γ belongs to $L^2(R^3, M^{1/\alpha}\bar{q}\, dv)$, and if $gM^{1/(2\alpha)}/\bar{q}^{1/2}$ is bounded, then $\gamma M^{1/(2\alpha)}\bar{q}^{1/2}$ is bounded.

Proof. Consider the operator

$$\hat{A}^{\alpha} = \bar{q}^{-1/2} M^{1/(2\alpha)} \bar{A} M^{-1/(2\alpha)} \bar{q}^{-1/2} = \hat{K}^{\alpha} - I \tag{A.3}$$

Then Eq. (A.2) is equivalent to

$$(\hat{K}^{\alpha} - I)\hat{\gamma}^{\alpha} = \hat{g}^{\alpha} \tag{A.4}$$

where

$$\hat{\gamma}^{\alpha} = M^{1/(2\alpha)} \bar{q}^{1/2} \gamma, \qquad \hat{g}^{\alpha} = M^{1/(2\alpha)} \bar{q}^{-1/2} g$$

For every $\alpha \geqslant 1$, the kernel $\hat{k}^{\alpha}(v, v')$ of $\hat{K}^{\alpha}$ is bounded pointwise by a constant, depending on α, $\bar{\rho}$ and $\bar{T}$, times $k_2(v(\alpha\bar{T})^{-1/2}, v'(\alpha\bar{T})^{-1/2})$, where $k_2(v, v')$ is the kernel defined in (48) of ref. 20. In Ref. 20, pp. 46–49, Grad proves that the kernel $k_2^{(3)}$ of the third iterate of the integral operator corresponding to k_2 is square-integrable, so that the same holds for $\hat{k}^{\alpha}$. Thus, $(\hat{K}^{\alpha})^3$ is a Hilbert–Schmidt operator, which implies that $\hat{K}^{\alpha}$ is compact on $L^2(dv)$. The kernel of $(\hat{K}^{\alpha} - I)$ is the linear span of $M^{1/(2\alpha)}\bar{q}^{1/2}$, while the kernel of $(\hat{K}^{\alpha} - I)^*$ is the linear span of $M^{1-1/(2\alpha)}\bar{q}^{1/2}$. Then, for every $\hat{g}^{\alpha} \in L^2(dv)$ such that $\int dv\, \hat{g}^{\alpha}(v)\, M^{1-1/(2\alpha)}(v)\, \bar{q}^{1/2}(v) = 0$, (A.4) has a solution in $L^2(dv)$, unique up to addition of a constant times $M^{1/(2\alpha)}\bar{q}^{1/2}$. Therefore, for every $\alpha \geqslant 1$, for every $g \in L^2(R^3, M^{1/\alpha}\bar{q}^{-1}\, dv)$ such that (A.1) holds, (A.2) has a solution in $L^2(R^3, M^{1/\alpha}\bar{q}\, dv)$; since g belongs to $L^2(R^3, M\bar{q}^{-1}\, dv)$ as well, the solution is unique, up to an additive constant, in $L^2(R^3, M\bar{q}\, dv)$. Moreover,

$$\sup_v \int dv'\, k_2(v, v')^2 < \infty \tag{A.5}$$

[ref. 20, (59) and (61)], so that, for any solution $\hat{\gamma}^{\alpha}$ of (A.4), $\|\hat{\gamma}^{\alpha}\|_{\infty}$ is bounded, up to a multiplicative constant, by

$$\|\hat{g}^{\alpha}\|_{\infty} + \|\hat{\gamma}^{\alpha}\|_{L^2}$$

which yields the last assertion of the statement. ∎

Proof of Theorem 3.1. The condition (3.6) is immediate from Theorem A.1. An explicit computation shows that the kernel $k(v, v')$ of $\bar{A} + \bar{q}I = K$ is given by a constant times

$$(|v - v'|)^{-1} \exp\{-[v' \cdot (v' - v)]^2/(2\bar{T}|v' - v|^2)\} \tag{A.6}$$

k is continuously differentiable in v, for $v \neq v'$, and $|\partial_v k(v, v')|$ is dominated, up to a multiplicative constant, by

$$k(v, v')(1 + 3|v'|^2/\bar{T} + |v|^2/\bar{T}) \tag{A.7}$$

As in the proof of Theorem A.1, for every $\alpha \geqslant 1$, $M^{1/(2\alpha)}(v)\, k(v, v')\, M^{-1/(2\alpha)}(v')$ is bounded by a constant times $k_2(v/(a\bar{T})^{1/2}, v'/(\alpha\bar{T})^{1/2})$, so that, by (59) and (61) in ref. 20, $M^{1/(2\alpha)}(v)\,|\partial_v k(v, \cdot)|\, M^{-1/(2\alpha)}(\cdot)$ is integrable and the integral is bounded in v. Since $|\phi|\, M^{1/(2\alpha)}$ is bounded for every $\alpha \geqslant 1$, this implies that $K\phi$ is continuously differentiable and $M^{1/(2\alpha)}\,|\partial(K\phi)|$ is bounded for every $\alpha \geqslant 1$, which yields the assertion, by the observation that $\phi = K\phi + v$. ▮

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